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LETTER TO THE EDITOR

Hamiltonian formalism for reversible non-equilibrium fluids with heat flow

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Abstract. Extended fluid dynamics, where, while the total entropy is conserved the particle one is not, is shown to have a canonical Hamiltonian structure in the space of Clebsch potentials and a non-canonical Hamiltonian structure in the space of physical variables.

In ideal compressible fluid dynamics there is no entropy exchange between the fluid particles. Thus, entropy is conserved and consequently the heat concept does not appear. This is expressed by the motion equation

$$-\sigma_{,t} = \operatorname{div}(\sigma \boldsymbol{v}) \tag{1}$$

where $\boldsymbol{v}$ is the fluid velocity and σ is the entropy density, $\sigma = \rho s$; ρ being the mass density and s the specific (=per unit mass) entropy.

Recently, Sieniutycz and Berry (SB) [1] proposed a generalisation of classical fluid dynamics based on extended thermodynamics of heat conducting fluids which are off, but not far off, Gibbs equilibrium. In addition to the basic variables of classical fluid dynamics $\{\boldsymbol{v}, \rho, s\}$, there appears a new variable, $\boldsymbol{j}$, the diffusive entropy flux; equation (1) changes into

$$-\sigma_{,t} = \operatorname{div}(\sigma \boldsymbol{v} + \boldsymbol{j}) \tag{2}$$

and the specific internal energy of the fluid now acquires $\boldsymbol{j}$ dependence:

$$e = e(\rho, \sigma, \boldsymbol{j}). \tag{3}$$

In the case of small $|\boldsymbol{j}|$, the first approximation to (3) is [1]:

$$e = e_0(\rho, \sigma) + g(\rho, \sigma)|\boldsymbol{j}|^2/2\rho^2 \tag{4}$$

and the total energy of the system becomes

$$E = \rho \frac{|\boldsymbol{v}|^2}{2} + \rho(e_0 + U) + g\frac{|\boldsymbol{j}|^2}{2\rho} \tag{5}$$

where $U = U(\boldsymbol{x})$ is an external potential. For the case of small $|\boldsymbol{j}|$, SB proposed a Hamilton principle for the extended fluid dynamics (EFD), based on the Lagrangian [1]

$$\mathscr{L} = L + \lambda[\rho_{,t} + \operatorname{div}(\rho \boldsymbol{v})] + \gamma[\sigma_{,t} + \operatorname{div}(\sigma \boldsymbol{v} + \boldsymbol{j})] \tag{6a}$$

$$L = \rho \frac{|\boldsymbol{v}|^2}{2} + g\frac{|\boldsymbol{j}|^2}{2\rho} - \rho(e_0 + U) \tag{6b}$$

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where λ and γ are Lagrange multipliers. (As shown in [1], (6*a*) is the Legendre transform of (5).)

In this letter I will show that the EFD is a Hamiltonian system and, also, that this conclusion remains true when one considers the case of arbitrary dependence of e on thermal variables and not only for the {small $|\boldsymbol{j}|$} case (4).

The Euler–Lagrange equations for the Lagrangian $\mathscr{L}$ (6) are:

$$\boldsymbol{v} = \boldsymbol{\nabla}(\lambda) + \rho^{-1}\sigma\boldsymbol{\nabla}(\gamma) \tag{7a}$$

$$\boldsymbol{j} = g^{-1}\rho\boldsymbol{\nabla}(\gamma) \tag{7b}$$

$$-\lambda_{,t} = (\rho e_0)_{,\rho} + U + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\lambda) - |\boldsymbol{v}|^2/2 - (g\rho^{-1})_{,\rho}|\boldsymbol{j}|^2/2 \tag{8a}$$

$$-\rho_{,t} = \operatorname{div}(\rho\boldsymbol{v}) \tag{8b}$$

$$-\gamma_{,t} = \rho e_{0,\sigma} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})(\gamma) - g_{,\sigma}\rho^{-1}|\boldsymbol{j}|^2/2 \tag{8c}$$

$$-\sigma_{,t} = \operatorname{div}(\sigma\boldsymbol{v} + \boldsymbol{j}). \tag{8d}$$

It is straightforward to check that (8) is a canonical Hamiltonian system: it can be written in the form

$$-(\lambda, \rho, \gamma, \sigma)'_{,t} = B^1(\delta h/\delta\lambda, \delta h/\delta\rho, \delta h/\delta\gamma, \delta h/\delta\sigma)' \tag{9}$$

where

$$B^1 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{\smash{\raisebox{.5em}{0}}} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{\smash{\raisebox{.5em}{0}}} & -1 & 0 \end{array}\right) \tag{10}$$

and h is the result of substitution into the total energy E (5) of $\boldsymbol{v}$ and $\boldsymbol{j}$ expressed through the constrained relations (7). Introducing

$$\mathbb{M} := \rho\boldsymbol{v} = \rho\boldsymbol{\nabla}(\lambda) + \sigma\boldsymbol{\nabla}(\gamma) \tag{11a}$$

$$\mathbb{J} := g\boldsymbol{j} = \rho\boldsymbol{\nabla}(\gamma) \tag{11b}$$

we can find, similar to the case of compressible fluid dynamics in [2], a non-canonical Hamiltonian structure in the space $\{\mathbb{M}, \mathbb{J}, \rho, \sigma\}$ of physical variables for which the Clebsch map (11) is a Hamiltonian (=canonical) map with respect to the canonical Hamiltonian structure (10): the motion equations in the physical space can be put into the form

$$-(\mathbb{M}, \mathbb{J}, \rho, \sigma)'_{,t} = B^2(\delta\bar{H}/\delta\mathbb{M}, \delta\bar{H}/\delta\mathbb{J}, \delta\bar{H}/\delta\rho, \delta\bar{H}/\delta\sigma)' \tag{12}$$

where

$$B^2 = \begin{array}{c} \\ M_\alpha \\ J_\alpha \\ \rho \\ \sigma \end{array}\!\!\begin{array}{c} \begin{array}{cccc} M_\beta & J_\beta & \rho & \sigma \end{array} \\ \begin{pmatrix} M_\beta\partial_\alpha + \partial_\beta M_\alpha & J_\beta\partial_\alpha + \partial_\beta J_\alpha & \rho\partial_\alpha & \sigma\partial_\alpha \\ J_\beta\partial_\alpha + \partial_\beta J_\alpha & 0 & 0 & \rho\partial_\alpha \\ \partial_\beta\rho & 0 & 0 & 0 \\ \partial_\beta\sigma & \partial_\beta\rho & 0 & 0 \end{pmatrix} \end{array}. \tag{13}$$

$\partial_\alpha = \partial/\partial x_\alpha$, where x_α, $1 \leqslant \alpha \leqslant m$, are Euclidean coordinates in $\mathbb{R}^m$, and

$$\bar{H} = \frac{|\mathbb{M}|^2}{2\rho} + \rho(e_0 + U) + \frac{|\mathbb{J}|^2}{2g\rho}. \tag{14}$$

The matrix B^2 (13) is Hamiltonian since it is naturally associated [3, chapter 8] with the dual space of the semidirect sum Lie algebra

$$\left[\begin{pmatrix} X^1 \\ Y^1 \\ a^1 \\ b^1 \end{pmatrix}, \begin{pmatrix} X^2 \\ Y^2 \\ a^2 \\ b^2 \end{pmatrix}\right] = \begin{pmatrix} [X^1, X^2] \\ [X^1, Y^2]-[X^2, Y^1] \\ X^1(a^2)-X^2(a^1)+\varepsilon Y^1(b^2)-\varepsilon Y^2(b^1) \\ X^1(b^2)-X^2(b^1) \end{pmatrix} \tag{15}$$

$$X^1, X^2, Y^1, Y^2 \in \mathscr{G} \qquad a^1, a^2, b^1, b^2 \in V$$

where $\mathscr{G}$ is a Lie algebra acting on V, and the matrix B^2 (13) is the particular case of (15) when $\varepsilon = 1$ and

$$\mathscr{G} = \{\text{vector fields on } \mathbb{R}^m\} \qquad V = C^\infty(\mathbb{R}^m). \tag{16}$$

Notice that if we exchange the entropy density variable σ for the specific entropy variable $s = \sigma/\rho$, the Hamiltonian matrix B^2 becomes

$$B^3 = \begin{array}{c} \\ \end{array}\begin{pmatrix} & M_\beta & J_\beta & \rho & s \\ M_\alpha & M_\beta\partial_\alpha + \partial_\beta M_\alpha & J_\beta\partial_\alpha + \partial_\beta J_\alpha & \rho\partial_\alpha & -s_{,\alpha} \\ J_\alpha & J_\beta\partial_\alpha + \partial_\beta J_\alpha & 0 & 0 & \rho\partial_\alpha\rho^{-1} \\ \rho & \partial_\beta\rho & 0 & 0 & 0 \\ s & \partial_\beta s & \rho^{-1}\partial_\beta\rho & 0 & 0 \end{pmatrix} \tag{17}$$

which is no longer linear in its variables: this is similar to the case of the superfluid irrotational helium-4 [4] but very dissimilar from the case of the standard fluid dynamics [2, 5, 6].

Finally, transforming the Hamiltonian matrix B^2 (13) into the space with the variables $\{\mathbb{M}, \boldsymbol{i}, \rho, \sigma\}$, where

$$\boldsymbol{i} = \frac{g}{\rho}\boldsymbol{j} = \mathbb{J}/\rho \tag{18}$$

is the thermal momentum, we obtain the affine Hamiltonian matrix

$$B^4 = \begin{pmatrix} & M_\beta & i_\beta & \rho & \sigma \\ M_\alpha & M_\beta\partial_\alpha + \partial_\beta M_\alpha & \partial_\beta i_\alpha - i_{\beta,\alpha} & \rho\partial_\alpha & \sigma\partial_\alpha \\ i_\alpha & i_\beta\partial_\alpha + i_{\alpha,\beta} & 0 & 0 & \varepsilon\partial_\alpha \\ \rho & \partial_\beta\rho & 0 & 0 & 0 \\ \sigma & \partial_\beta\sigma & \varepsilon\partial_\beta & 0 & 0 \end{pmatrix} \tag{19}$$

which also serves magnetohydrodynamics in the $\mathbb{A}$ representation [2, 7]. (The matrix B^4 is a particular one-dimensional Abelian case of the spin-glass Hamiltonian matrix in [8].) This matrix services the case of the most general non-equilibrium reversible fluids, when the total energy of the system is given by the formula

$$H = \frac{|\mathbb{M}|^2}{2\rho} + \rho U + \rho e(\rho, \sigma, \boldsymbol{i}) \tag{20}$$

where e, the specific internal energy, is an *arbitrary* function of ρ, σ and $\boldsymbol{i}$, with the Gibbs relation [1]

$$\mathrm{d}(\rho e) = \mu\, \mathrm{d}\rho + T\, \mathrm{d}\sigma + \boldsymbol{j}\cdot \mathrm{d}\boldsymbol{i} \tag{21}$$

with μ and T being non-equilibrium chemical potential and temperature, respectively. The motion equations of EFD, generated by the Hamiltonian matrix B^4 (19) and the Hamiltonian H (20) are:

$$-M_{\alpha,t} = (\Pi_{\alpha\beta} + \delta_{\alpha\beta}P)_{,\beta} + \rho U_{,\alpha} \tag{22a}$$

$$-i_{\alpha,t} = (\boldsymbol{v}\cdot\boldsymbol{i} + T)_{,\alpha} + v_\beta(i_{\alpha,\beta} - i_{\beta,\alpha}) \tag{22b}$$

$$-\rho_{,t} = \operatorname{div}(\rho\boldsymbol{v}) \tag{22c}$$

$$-\sigma_{,t} = \operatorname{div}(\sigma\boldsymbol{v} + \boldsymbol{j}) \tag{22d}$$

where

$$\Pi_{\alpha\beta} = \rho^{-1}M_\alpha M_\beta + i_\alpha j_\beta \tag{23}$$

is the stress tensor, and

$$P = \rho\mu + \sigma T - \rho e \tag{24}$$

is the pressure. In terms of the fluid velocity $\boldsymbol{v}$, one then has

$$-v_{\alpha,t} = v_\beta v_{\alpha,\beta} + \rho^{-1}P_{,\alpha} + U_{,\alpha} + \rho^{-1}(i_\alpha j_\beta)_{,\beta}. \tag{25}$$

The following remarks can be made.

(a) For the general EFD system (22), one also has a Hamilton principle with the Lagrangian $\mathscr{L}$ (6a), where the Lagrangian L is now given, instead of formula (6b), by the formula

$$L = \rho\frac{|\boldsymbol{v}|^2}{2} + \boldsymbol{j}\cdot\boldsymbol{i} - \rho(e+U) \tag{26}$$

and one still has the canonical Hamiltonian system (9), (10). These facts follow from the general recipes in [7] applied to the intermediate Hamiltonian matrix (13), or can be easily checked directly.

(b) Deleting the $\boldsymbol{i}$ (respectively $\mathbb{J}$) row and columns from the matrix B^4 (19) (respectively B^2 (13)) one gets the Hamiltonian matrix of classical fluid dynamics. The inverse procedure, of $\boldsymbol{i}$ (respectively $\mathbb{J}$) extensions of numerous Hamiltonian matrices of ideal continuous systems, allows one to incorporate reversible non-equilibrium processes into a great variety of known continuous Hamiltonian systems, all of which neglect diffusion of entropy (and other quantities).

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